Different Traces of Quantum Systems Having the Same Classical Limit

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Abstract

Many quantum systems may have the same classical limit. We argue that in the classical limit their traces do not necessarily converge one to another. The trace formula allows to express quantum traces by means of classical quantities as sums over periodic orbits of the classical system. To explain the lack of convergence of the traces we need the quantum corrections to the classical actions of periodic orbits. The four versions of the quantum baker map on the sphere serve as an illustration of this problem.

PACS: 05.45.Mt

Keywords: Trace formula, Q-representation, Baker map

1. Introduction

The analysis of a quantum system in the semiclassical regime allows one to approximate quantum traces by the sum over periodic orbits of the corresponding classical system. This relation is provided by the famous Gutzwiller Trace Formula [1] which establishes a fruitful link between a quantum system and its classical counterpart. It is well known that infinitely many quantum systems may have the same classical limit. A natural problem arises how different such quantum systems might be. Let us concentrate on quantum traces which allow us to compute the spectrum. The Gutzwiller-like formula approximates in the semiclassical limit the traces of the evolution operator U by

$$\mathrm{Tr}\, U^n = \sum_{\mathrm{p.o.}} A_o \cdot e^{iS_o/\hbar} \; . \qquad (1)$$

The sum is taken over all periodic orbits with periods equal or less than n. The orbits contribute with amplitudes A_o which depend on their stability while the phases are equal to their classical actions (measured in $\hbar$ units). Possible Maslov indices, not mentioned in the formula (1), do not influence the following argument since they are related to the classical system. An example of two quantum systems, with the same classical limit, having the trace of the Floquet operator constant with respect to $\hbar$ and equal to 0 and $\sqrt{2}$ respectively is provided by the quantum baker map on the sphere [2]. To approximate these traces semi-classically in the spirit of (1) $\hbar$ corrections to the classical actions, $S_o = S_{\mathrm{class}} + \hbar S_{\mathrm{quant}}$, are needed. These corrections disappear in the classical limit $\hbar \to 0$, but they result in corrections of order unity to the traces. The traces of such quantum systems can not be expressed using only classical quantities and may not converge in the classical limit. Many quantum models for which the semiclassical treatment is known [3–9] do not seem to suffer from this problem — their traces do converge to those predicted by the sum over periodic orbits without any quantum corrections. An example where such $\hbar$ corrections are needed in the semiclassical analysis is the original version of the kicked top [10]. On the other hand a small change of this quantum model [8] is sufficient to eliminate this problem.

The Fourier transform with respect to $1/\hbar$ of the quantity $\langle \Psi_0 | U^n | \Psi_0 \rangle$ for some state $|\Psi_0\rangle$ may be used to check the semiclassical predictions. The modulus of this transform has peaks at the classical actions of orbits of n iterations of the map, if the state $|\Psi_0\rangle$ overlaps with the state localized on the periodic orbit. The phase of the transform will contain Maslov indices and first order quantum corrections to the actions of periodic orbits. Contributions stemming from periodic points with the same classical action form single peak in the transform. In this work we propose a way to analyze the classical action and its quantum correction coming from a single periodic point of n iterations of the map. The paper is organized as follows. The decomposition of the

Husimi representation of the evolution operator into a sum over periodic orbit is presented in section 2. The analysis of actions of periodic orbits and their corrections found for the quantum models of the baker map on the sphere is performed in section 3. Conclusions and consequences for the spectra are discussed in section 4.

2. Coherent State Decomposition of the Evolution Operator

Lets Ω be a classical phase space. Consider a classical area-preserving map $\Theta : \Omega \to \Omega$ and a corresponding quantum operator U acting on a Hilbert space $\mathcal{H}$. Assume we have a family of generalized coherent states $|\alpha\rangle \in \mathcal{H}$ for any $\alpha \in \Omega$ fulfilling the minimal uncertainty relation. Such a family of states is found for several examples of classical phase spaces. The coherent states form an overcomplete basis allowing the decomposition of the identity operator $\int_\Omega |\alpha\rangle\langle\alpha|d\alpha = \mathbf{1}$. We introduce the generalized Husimi [11] representation of an operator $\mathcal{A}$

$$H_{\mathcal{A}}(\alpha) = \langle\alpha|\mathcal{A}|\alpha\rangle. \tag{2}$$

If $\mathcal{A}$ is a projection operator onto state $|\Psi\rangle$, it reduces to the standard Q-representation of a pure state $H_{|\Psi\rangle\langle\Psi|}(\alpha) = |\langle\alpha|\Psi\rangle|^2$. Representing the identity operator as a sum over coherent states we have

$$\int_\Omega H_{\mathcal{A}}(\alpha)d\alpha = \mathrm{Tr}\,\mathcal{A}, \quad \int_\Omega H_{|\Psi\rangle\langle\Psi|}(\alpha)d\alpha = 1. \tag{3}$$

The family of coherent states allows us to establish a link between classical and quantum dynamics [12,13]. If for any $\rho > 0$

$$\lim_{\hbar\to 0} \inf_{\alpha\in\Omega} \int_{C(\Theta(\alpha),\rho)} |\langle\alpha'|U|\alpha\rangle|^2 d\alpha' = 1 , \tag{4}$$

where $C(\Theta(\alpha),\rho)$ denotes a circle centered at $\Theta(\alpha)$ with the radius ρ, we say that the quantization of Θ into U is *regular* with respect to the family of coherent states $|\alpha\rangle$. The formula (4) means that $|\langle\alpha'|U|\alpha\rangle|^2$ tends to $\delta\left(\alpha' - \Theta(\alpha)\right)$ in the classical limit, so the quantum state $U|\alpha\rangle$ is then concentrated in the vicinity of the classical image $\Theta(\alpha)$.

Let us consider the Husimi representation of the evolution operator $H_{U^n}(\alpha) = \langle\alpha|U^n|\alpha\rangle$. The above argument shows that in the semiclassical regime the state $U^n|\alpha\rangle$ is concentrated at $\Theta^n(\alpha)$ and is close to 0 everywhere else. Therefore $H_{U^n}(\alpha)$ is localized on the periodic points of classical map ($\Theta^n(\alpha) = \alpha$). This useful feature was observed in [14] and served to demonstrate classical to quantum correspondence. The author of [14] investigated only the squared modulus of the function $H_{U^n}(\alpha)$.

We know that $\mathrm{Tr}\,U^n = \int_\Omega \langle\alpha|U^n|\alpha\rangle d\alpha$ and the integrated function is localized on the periodic points in the classical limit ($\hbar \to 0$). This fact allows us to decompose the quantum traces into sums over periodic orbits for any quantum maps, under the standard assumption that all periodic points are isolated, but without employing the saddle point approximation. The modulus of the integrated overlap $\langle\alpha|U^n|\alpha\rangle$ near the periodic point corresponds to the amplitude A_o in equation (1) and depends on the stability of this periodic orbit. The phase of the function $H_{U^n}(\alpha)$ seems to have a stationary point at the periodic orbit as observed in [10] for the quantum kicked top and reported later on in this work for the baker map on the sphere.

3. Various Quantizations of Baker Map on the Sphere

Quantizing the baker map on the sphere [2] we define four different quantum maps B_k ($k = 0 \ldots 3$), corresponding to the same classical system. In spite of this fact, they have different traces and spectra. By a quantum map on the sphere we understand a $N = 2j + 1$ dimensional unitary operator acting in the Hilbert space of angular momentum, where $j(j + 1)$ is the eigenvalue of $\hat{J}^2$ operator which is conserved by the evolution. The four unitary operators B_k may be expressed in the eigenbasis $|j, m\rangle$ of $\hat{J}_z$ operator

$$B_{(ab)} = R^{-1} \left[\begin{array}{cc} R'^{(a)} & 0 \\ 0 & R''^{(b)} \end{array} \right] , \tag{5}$$

where index (ab) with $a, b = 0, 1$ corresponds to binary representation of k ($B_{(00)} = B_0$, etc.), R is the Wigner rotation matrix $R_{m',m} = \langle j, m|e^{-i\frac{\pi}{2}\hat{J}_y}|j, m'\rangle$ and the matrices R' and R'' are constructed by choosing odd or even columns

of the matrix R in the following way

$$R'^{(a)}_{m,l} := \sqrt{2}R_{m,2l+j+a}, \quad m,l = -j\ldots-\frac{1}{2};$$

$$R''^{(b)}_{m,l} := \sqrt{2}R_{m,2l-j-1+b}, \quad m,l = \frac{1}{2}\ldots j. \qquad (6)$$

This construction requires the dimension N to be an even integer so the quantum number j is half integer. The volume of the classical phase space is equal to 4π (the unit sphere), so $N = \frac{4\pi}{2\pi\hbar}$, which relates $\hbar$ to the size of the Hilbert space ($\hbar = 2/N$) and the classical limit corresponds to $N \to \infty$. Now the coherent state $|\alpha\rangle$ will refer to a $SU(2)$ vector coherent state $|\theta, \phi\rangle$ [15,16].

In figure 1 we plot the Husimi representation of the operator $(B_1)^3$ for $N = 100$, the coor-
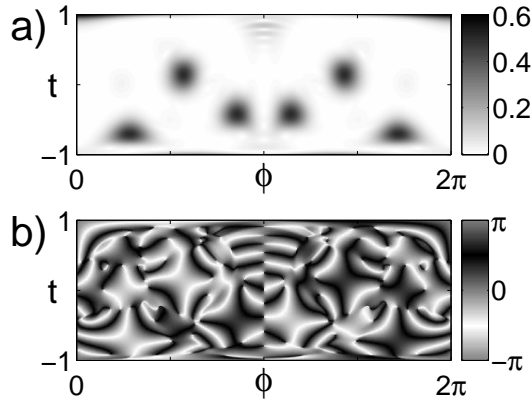

Figure 1. The Husimi representation of three step propagator for the baker map on the sphere is plotted using gray scale (the coordinates are azimuth angle ϕ and $t = \cos\theta$), a) represents the absolute value and b) the phase of the function $\langle\alpha|(B_1)^3|\alpha\rangle$. Note the stationary phase at the points where the modulus of the function is maximal.

dinates are the azimuth angle ϕ and $t = \cos\theta$. The complex function $\langle\alpha|(B_1)^3|\alpha\rangle$ is well localized near the periodic points of three iterations of the classical baker map on the sphere. The analogous plots for other versions of the model look almost the same. We want to investigate the phase of contribution coming from single periodic point. The numerical integration performed for several dimensions N shows that the phases of such contributions are very well approximated

by $\arg(\langle\alpha_o|(B_k)^n|\alpha_o\rangle)$, where $|\alpha_o\rangle$ is the coherent state centered at the periodic point. This is so due to the stationarity of the phase of the function $\langle\alpha|(B_k)^n|\alpha\rangle$ at the periodic point, as pointed out before.

We have calculated the phase of the Husimi representation of $(B_k)^3$ for four versions of the model defined by equations (5) and (6) at the periodic point $\theta = \arccos\left(-\frac{5}{7}\right)$, $\phi = \frac{2}{7}\pi$. The results are plotted in figure 2 where the four different symbols denote the four different quantum
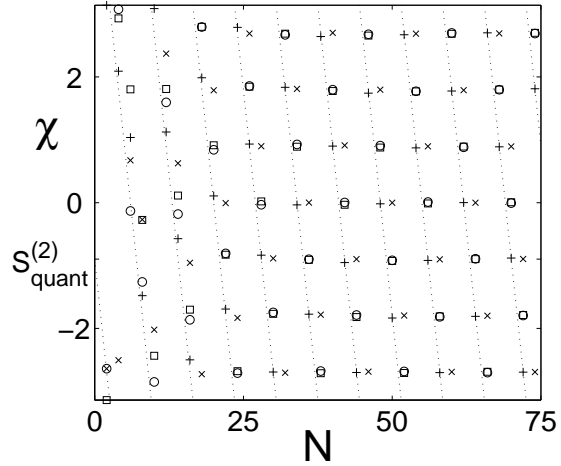

Figure 2. The phase $\chi_k = \arg(\langle\alpha_o|(B_k)^3|\alpha_o\rangle)$ for the baker map on the sphere is plotted as a function of $N = 2/\hbar$ the dimension of Hilbert space. The four versions of the model $k = 0\ldots3$ are marked by $\circ, \times, +, \square$ respectively. Note the linear behavior for larger value of N, the dotted line ($\chi = -\frac{2}{7}\pi N - \frac{2}{7}\pi$) represents the asymptotic behavior of one of the models — B_2 (+). The quantum correction $S^{(2)}_{\text{quant}}$ corresponds to the point, where this line crosses the $N = 0$ axes. The coherent state $|\alpha_o\rangle$ is chosen at the periodic point $\theta = \arccos\left(-\frac{5}{7}\right)$, $\phi = \frac{2}{7}\pi$.

maps. The phase χ is plotted in the interval $(-\pi, \pi]$. We observe that, starting from some dimension, the phase depends linearly on N modulo 2π. The points for four versions of the model, marked with different symbols, form parallel lines and they do not converge one to another. The phase $\chi_k = \arg(\langle\alpha_o|(B_k)^3|\alpha_o\rangle)$ can be then well parametrized by $S_{\text{class}}\frac{N}{2} + S^{(k)}_{\text{quant}}$. The slope of these lines is the same and corresponds to the

classical action of the orbit S_{class} equal to $-\frac{2}{7}2\pi$ (only the fractional part of 2π is important here). The quantum corrections S_{quant} to the actions of periodic orbits may be found by an extrapolation of the linear behavior to $N = 0$, as it is demonstrated in figure 2 for $k = 2$. The asymptotic dotted line crosses $N = 0$ axes at $\chi = S_{\text{quant}}^{(2)} = -\frac{2}{7}\pi$. The quantum corrections for other quantizations are $S_{\text{quant}}^{(0)} = S_{\text{quant}}^{(3)} = 0$ and $S_{\text{quant}}^{(1)} = \frac{2}{7}\pi$.

The investigation performed for other periodic points shows that the quantum corrections S_{quant} are the same for periodic points belonging to the same periodic orbit and are equal to $p \times S_{\text{quant}}$(primary orbit) if the periodic orbit is p repetitions of a primary orbit, therefore with this respect they behave like classical actions.

4. Conclusions

In this paper we have focussed on the semiclassical behavior of quantum systems having the same classical limit. The corrections to the quantum traces, not disappearing in the classical limit, arise from $\hbar$ corrections to the actions of periodic orbits which do disappear in the classical limit. It means that the traces of a quantum system having such corrections may not be approximated by the trace formula constructed on purely classical properties of the system. For any quantum system we propose a way to find the contributions to the traces coming from single periodic points of n iterations of the classical map by means of autocorrelation function $\langle \alpha | U^n | \alpha \rangle$. This function is localized at the periodic orbits of classical system (U stands for evolution operator and $|\alpha\rangle$ denotes the coherent state – the wave packet). The four versions of the quantum baker map on the sphere serve as an example of such behavior. We investigate the phase $\chi = \arg(\langle \alpha_o | B^n | \alpha_o \rangle)$ for the coherent state $|\alpha_o\rangle$ pointing at the periodic points of the classical baker map on the sphere. We have observed the linear behavior of χ as a function of $1/\hbar$ in semiclassical regime.

We believe that quantum systems having the same classical limit may have different traces not converging one to another in the classical limit. These traces can be explained by means of quantum corrections (first order in $\hbar$) to the actions of periodic orbits in the semiclassical approxima-

tion. The spectra of the Floquet operators of such systems contain $N \sim 1/\hbar$ eigenvalues placed on the unit circle. They may be expressed in terms of the traces. The difference of quantum traces of order of 1 requires difference between eigenvalues of different quantum systems at least of order of $1/N$, i.e. the same as the mean level spacing. The arguments presented in this paper should be valid even in the case of infinite Hilbert space (e.g. for continuous dynamics).

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